

# Puzzles in Time Delay and Fermat Principle in Gravitational Lensing

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## ABSTRACT

The current standard time delay formula (CSTD) in gravitational lensing and its claimed relation to the lens equation through Fermat’s principle (least time principle) have been puzzling to the author for some time. We find that the so-called geometric path difference term of the CSTD is an error, and it causes a double counting of the correct time delay. We examined the deflection angle and the time delay of a photon trajectory in the Schwarzschild metric that allows exact perturbative calculations in the gravitational parameter  $GM$  in two coordinate systems – the standard Schwarzschild coordinate system and the isotropic Schwarzschild coordinate system. We identify a coordinate dependent term in the time delay which becomes irrelevant for the arrival time difference of two images. It deems necessary to sort out unambiguously what is what we measure. We calculate the second order corrections for the deflection angle and time delay. The CSTD does generate correct lens equations including multiple scattering lens equations under the variations and may be best understood as a generating function. It is presently unclear what the significance is. We call to reanalyze the existing strong lensing data with time delays.

**Key words:** gravitational lensing

## 1 INTRODUCTION

Shapiro (1964) championed the time delay of the radar echo off of the inner planets as a fourth test of Einstein general relativity. The so-called classic three tests were suggested by Einstein and they are Mercury precession, astrometric shifts of the background stars due to the gravitational lensing by the Sun, and gravitational redshifts (Pound and Rebka 1959). In the same year, Refsdal (1964b) suggested to measure the time delay difference (or arrival time difference) between two images to use in conjunction with other measurables (position and flux of the images) of supernovae to determine the Hubble constant and mass of the lens galaxies. The time delay measurement is a central issue in strong gravitational lensing where multiple images are detected, especially because the time data can help break the degeneracy of the lens fitting models. There have been monitoring campaigns of multiple image quasars in radio and optical (Vuissoz et al 2008; Coles 2008). With large high cadence survey telescopes in plan (LSST, WFIRST, EUCLID) time delay measurements will become a routine business, and it is expected to see many multiply imaged supernovae (Kirkby et al. 2006).

The time delay function (2d scalar function) is also

known to generate the lens equation (2d vector equation) through the stationarity or extremum condition (commonly referred to as the Fermat’s principle) for single scattering lenses (Schneider 1985; Read et al 2007) and multiple scattering lenses withal (Blandford and Narayan 1986). The single scattering lens equations are well known from direct derivations of the photon path in the Schwarzschild metric and its generalizations in the effectively Newtonian gravitational field. The multiple scattering lens equation can also be derived directly by looking at the photon path in 3-space (Rhie and Bennett 2011) and it confirms the validity of the variational method.

The current standard time delay function is made of the geometric path difference term and gravitational potential term. The so-called geometric path difference term of the CSTD is essentially a quadratic function of the position difference between the source and image, and we have been puzzled by it for some time even though it is widely used unsuspected and being incorporated into pipeline codes for systematic studies of gravitational lensing (Vuissoz et al (2008), Coles (2008), Read et al (2007), and references therein). We examine the CSTD for a Schwarzschild black hole lens which has the virtue of allowing exact calculations of the time delays and find the CSTD wrong. It effectively doubles the correct time delay. We trace the origin of the current standard time delay formula form to Cooke and Kan-

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towski (1975) (CK75 hereafter) and find that the so-called geometric path difference term suffers from mistakes. CK75 uses the so-called isotropic metric (in the linear approximation). Thus we discuss the deflection angle and time delay of a photon path in the Schwarzschild metric in two coordinate systems – the standard Schwarzschild coordinates and the isotropic Schwarzschild coordinates, where the latter can be obtained from the former (or vice versa) by changing the radial coordinates. In the linear order in  $r_s \equiv 2GM$  where  $M$  is the mass of the Schwarzschild black hole, the radial coordinates only differ by a constant:  $r = u + r_s/2$  where  $r$  and  $u$  are the standard and isotropic radial coordinates respectively. We will see that the two metric forms (both asymptotically flat) result in the same time delay difference for two images. At the same time, it leaves a question as to what is the time the observer measures. We examine the deflection angle and find that the difference between the coordinate systems is effectively of the second order. In other words, the deflection angles measured in the two coordinate systems are the same while the time delays are different. We calculate the second order corrections to the deflection angle and time delay. We conclude in section 6.

### 2 THE CURRENT STANDARD TIME DELAY FORMULA

The current standard time delay formula used in the lensing community can be found, for example, in eq.(A1) of Read et al (2007) which is reproduced here for convenience.

$$(1+z_L) \left( \frac{D_L^2}{D} \frac{(\theta_I - \theta_s)^2}{2} - \frac{4G}{c^2} \int \Sigma(\theta') \ln |\theta_I - \theta'| d^2\theta' \right). \quad (1)$$

The first term is referred to as the time delay due to the geometric path length difference between the deflected and undeflected paths, and the second term is referred to as the time delay due to the gravitational potential acting on the photon along the path.

For a Schwarzschild black hole of mass  $M$ , eq.(1) becomes

$$ct(\theta_I; \theta_s) = (1+z_L) \left( \frac{D_L^2}{D} \frac{(\theta_I - \theta_s)^2}{2} - \frac{4GM}{c^2} \ln |\theta_I - \theta_L| \right) \quad (2)$$

where  $D_L$  is the distance (from the observer) to the lens and  $D$  is the reduced distance.  $\theta$  is the angular position variable, and the subscripts  $s, L, I$  denote source, lens, and image respectively.  $z_L$  is the redshift of the lens. ( $c$  is the speed of light, which will be set to be 1 hereafter:  $c = 1$ .) We may say that the time function is made of a “quadratic term” and a logarithmic term. The stationary condition with respect to the variation of the image position,  $dt/d\theta = 0$ , generates the lens equation. Since  $\theta$  is a two-dimensional angular variable in general, we take it as a vector, and the resulting equation is the well-known single lens equation.

$$0 = (\theta_I - \theta_s) - \frac{\theta_E^2 (\theta_I - \theta_L)}{|\theta_I - \theta_L|^2}; \quad \theta_E^2 \equiv \frac{4GMD}{D_L^2} \quad (3)$$

where  $\theta_E$  is the angular Einstein radius. This stationarity condition is commonly referred to as Fermat’s principle: an image forms such that the time is an extremum (even though Fermat’s principle for Snell’s law invokes the notion of the

least time.) The single point lens equation (3) has two images whose positions are collinear as is well known.

$$\theta_{I\pm} = \frac{\theta_s}{2} \left( 1 \pm \left( 1 + \frac{4\theta_E^2}{\theta_s^2} \right)^{1/2} \right) \quad (4)$$

$\theta_L$  has been set to zero by translating the coordinate system. The arrival time difference ( $t(\theta_-) - t(\theta_+)$ :  $\theta_-$  is the position of the dimmer image and arrives later) due to the quadratic term and the logarithmic term are

$$\Delta t_{12}(quad) = (1+z_L)2GMs(s^2+4)^{1/2}; \quad s^2 \equiv \frac{\theta_s^2}{\theta_E^2} \quad (5)$$

$$\Delta t_{12}(log) = (1+z_L)8GM \ln \left( \frac{s}{2} + \left( \frac{s^2}{4} + 1 \right)^{1/2} \right). \quad (6)$$

For  $s \lesssim 1$ , the both become  $\Delta t_{images} \approx (1+z_L)4GMs$ . The total time delay between two images would be  $\approx (1+z_L)8GMs$ .

#### 2.1 Refsdal’s Time Delay

Refsdal (1964b) shows that the arrival time difference of the two images of a spherical galaxy lens is  $\approx 8GMs$  in eq.(6-refsdal). The time delay formula was derived in eq.(30-refsdal) of Refsdal (1964a), and we may write it as follows.

$$\Delta t_{12} = \int_0^s \frac{D_s}{D_2} (\theta_1 - \theta_2) ds = \frac{D_s}{D_2} \frac{R_E^2}{D_1} \int_0^s (s^2 + 4)^{1/2} ds \quad (7)$$

where  $D_s = D_1 + D_s$  and  $s$  is the dimensionless source position variable. Refsdal didn’t present the exact integral, but it is easy to do.

$$\Delta t_{12} = \frac{R_E^2}{D} \left( \frac{1}{2} (b_+^2 - b_-^2) + \ln b_+ \right); \quad b_+ \equiv \frac{s}{2} + \left( \frac{s^2}{4} + 1 \right)^{1/2}. \quad (8)$$

$b_+$  is nothing but the position of the positive image (in units of the Einstein ring radius). The negative image position  $b_- = s - b_+$  and  $b_- = b_+^{-1}$ . Thus,

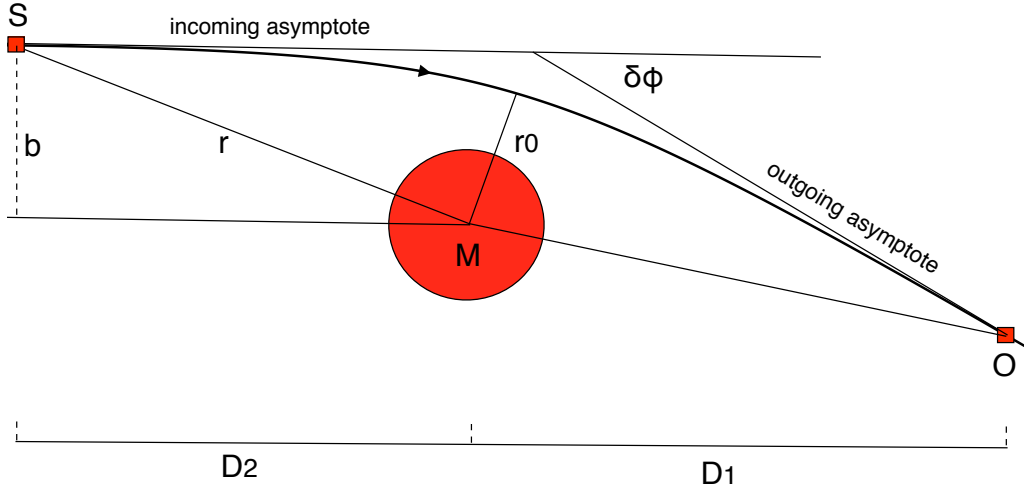
$$\Delta t_{12} = \frac{R_E^2}{D} \left( \frac{1}{2} ((b_- - s)^2 - (b_+ - s)^2) - \ln \left| \frac{b_-}{b_+} \right| \right). \quad (9)$$

It is a surprise; it is exactly the CSTD at  $z_L = 0$ !! Unfortunately, we have no clue how to understand the definition of the time delay in eq.(7): how integrating the angular separation of the two images as the source moves from  $s = 0$  (where there is no arrival time difference between the two images) to  $s = s$  is supposed to correspond to the time delay. Refsdal doesn’t explain it. It must have been self-evident to him. In short, we don’t know how he got it wrong.

#### 2.2 Simply Looking at Fig.1

Figure 1 shows a deflected photon path in  $(r, \phi)$  scattering plane. If we look at fig.1, the path difference between the actual photon path and the would-be straight line is  $\Delta \approx r_0 \delta\phi = 2r_s$ . It is a constant irrelevantly of the path.

We can try to do better. Let’s assign viewing angles by the observer of the image, source, and lens ( $M$ ) as  $\alpha, \beta$ , and  $\gamma$  respectively, then we may estimate it as  $\Delta \approx r_1(\alpha - \gamma)\delta\phi$ . Or, we may estimate it a bit more carefully by looking at the outgoing and incoming paths separately:  $\Delta_1 \approx r_1(\alpha - \gamma)^2$  and  $\Delta_2 \approx r_2(r_1/r_2)^2(\alpha - \gamma)^2$ . Then



**Figure 1.** Depiction of a photon path (arrowed thick curve), from a source  $S$  to an observer  $O$ , passing by a spherically symmetric mass  $M$  in Schwarzschild coordinates.  $r_0$  is the distance of the closest approach,  $\delta\phi$  is the scattering angle, and  $b$  is the impact distance. In the small scattering approximation relevant in most of the astronomical practices,  $\delta\phi \ll 1$ ,  $b \approx r_0$  and  $\sqrt{r^2 - r_0^2}$  is the path length of the photon from  $S$  to the center of the mass  $M$  in the absence of the gravitational interaction.

$$\Delta = \Delta_1 + \Delta_2 \approx \frac{D_1^2}{D} (\alpha - \gamma)^2. \quad (10)$$

The path length difference is larger for the brighter image. In other words, the arrival time difference of the two images will be opposite to what is implied in the “quadratic term.” The consequence is that the arrival time difference between the two images will vanish because the magnitudes of the “quadratic term” and logarithmic term are practically equal as we have seen in the previous section. The lens equation (the relation between the source and image positions) can be found easily.

$$\alpha - \beta = \frac{D}{D_1^2} \delta\phi = \frac{R_E^2}{D_1^3 (\alpha - \gamma)}. \quad (11)$$

Now we can express the geometric path difference  $\Delta$  in terms of  $(\alpha - \beta)$ .

$$\Delta \approx \frac{2r_s R_E^2}{D_1^2 (\alpha - \beta)^2}. \quad (12)$$

It may be best called an “inverse quadratic term” than “quadratic term.”

### 2.3 Gott and Gunn, Cooke and Kantowski, and Schneider’s Fermat Principle

Gott and Gunn (1974) wrote down without explanation the quadratic part of Refsdal’s time delay in eq.(9) as the correct time delay between two images. It is hard to know the underlying reasoning.

Cooke and Kantowski (1975) (CK75 hereafter) clearly lays out a reasoning for their calculation and hence is of interest here. CK75 interpreted the “quadratic term” of Gott and Gunn as due to the geometric path difference and purported to “complete” the time delay formula leading to the CSTD for an arbitrary mass distribution where the weak gravity approximation (linear in  $GM$ ) holds. CK75 doesn’t

mention that the CSTD was obtained by Refsdal for a spherically symmetric mass even though they cite Refsdal’s 1964a and 1964b. It is possible that CK75 didn’t know it, and they don’t mention what might be the underlying reason of Refsdal’s definition of the time delay perhaps because of it. CK75 does offer their own reasoning for the “quadratic term” as has been stated already and lays out a rule for the linear calculation. The “quadratic term” they obtain is of the second order in  $GM$ , hence it is a violation of their first order approximation assumption. Their calculation of the “quadratic term” amounts to a confusion where  $\gamma$  in eq.(10) is replaced by  $\beta$ , which led to the “quadratic form” instead of the “inverse quadratic form.” The difference of factor 2 comes from the slight difference in the definition of the path difference, (See Fig.2 of CK75.), which doesn’t concern us especially because the “quadratic term” will be shown to be irrelevant to the time delay.

Schneider (1985) “improved” CK75’s CSTD and introduced a variational principle to derive the lens equation from the CSTD time delay. Schneider called it Fermat’s Principle. Once the CSTD is found invalid as a measure of the flight time of a photon, the notion of Fermat’s principle may be best abandoned.

## 3 TIME DELAY DUE TO A SPHERICALLY SYMMETRIC MASS

### 3.1 The Standard Schwarzschild Metric

The deflection angle of a photon trajectory due to a spherically symmetric mass  $M$  and its flight duration can be calculated exactly by solving the equation of motion in the following standard Schwarzschild metric.

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2; \quad A(r)^{-1} = B(r) = 1 - \frac{2GM}{r}. \quad (13)$$

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The equation of motion can be derived from the variational principle,  $0 = \delta \int ds$ .

$$0 = \frac{d^2 x^\mu}{dp^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{dp} \frac{dx^\beta}{dp} \quad (14)$$

where  $p$  is the line parameter and  $\Gamma_{\alpha\beta}^\mu$  is the Christoffel symbol that is a function of the linear derivatives of the metric components.

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\lambda} (\partial_\mu g_{\nu\lambda} \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \quad (15)$$

where  $g^{\mu\lambda}$  is the inverse metric components:  $g^{\mu\lambda} g_{\lambda\alpha} = \delta_\alpha^\mu$ . Because of the spherical symmetry, the photon path lies on a plane, say  $\theta = \pi/2$ , and the equations involve three variables:  $t$ ,  $r$ , and  $\phi$ .

$$\frac{dt}{dp} = \frac{1}{B}; \quad \frac{d\phi}{dp} = \frac{J}{r^2}; \quad A \left( \frac{dr}{dp} \right)^2 = \frac{1}{B} - \frac{J^2}{r^2}. \quad (16)$$

where  $J$  is the angular momentum. From the equations, one can get  $d\phi = g(r)dr$  or  $dt = f(r)dr$ . The scattering (or deflection) angle is obtained by integrating  $d\phi = g(r)dr$ .

$$\int d\phi = \left( \int_{r_2}^{r_0} + \int_{r_0}^{r_1} \right) \frac{\sqrt{A/r} \, dr}{\sqrt{\frac{B_0 r^2}{Br_0^2} - 1}} \quad (17)$$

The integration  $\int d\phi = \int g(r)dr$  doesn't relent to a nice formula unlike in Newtonian but can be done easily by expanding in  $GM$  (weak gravity approximation), which is suitable for astrophysical purposes. The scattering angle comes out to be  $\delta\phi = 4GM/r_0$  as is well known, and the weak gravity approximation is also commonly called the small angle approximation.  $r_0$  is the distance of the closest approach of the photon path to the mass  $M$  as is shown in figure 1. In order to obtain the the Schwarzschild coordinate time duration it takes for the photon to travel from the source to the observer, one can integrate  $dt = f(r)dr$  in the same small angle approximation.

$$\int dt = \left( \int_{r_2}^{r_0} + \int_{r_0}^{r_1} \right) \frac{\sqrt{A/B} \, dr}{\sqrt{1 - \frac{Br_0^2}{B_0 r^2}}} \quad (18)$$

where  $B_0 = B(r_0)$ . The duration either from source at  $r = r_2$  to  $r_0$  or from  $r_0$  to the observer at  $r = r_1$  is given as follows. (See, for example, eq.(8.7.4) of Weinberg (1972); the speed of light  $c = 1$ .)

$$t(r, r_0) = \sqrt{r^2 - r_0^2} + 2GM \ln \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + GM \left( \frac{r - r_0}{r + r_0} \right)^{1/2} + \mathcal{O}(r_s^2/r_0^2). \quad (19)$$

where  $r_s = 2GM$  is the Schwarzschild radius. The Shapiro time delay formula of the radar echo eq.(1-shapiro) of Shapiro (1964) can be obtained from eq.(19).

Note that the first term does not depend on the mass  $M$  and should be the time the photon takes to travel from the source to the center of the mass  $M$  when the gravitation is turned off. It can be directly calculated from eq.(18) by setting  $A = B = 1$  and should also be intuitively evident from figure 1. The proper time duration measured by the

observer is obtained by multiplying the time dilation factor  $B(r)^{1/2}$  in eq.(13) to the Schwarzschild coordinate time (e.g., eq.(6.3.46) of Wald (1984)). Thus, the total proper time delay due to the general relativity in the linear order in  $r_s/r_0$  is

$$\Delta\tau/r_s = \ln \left( \frac{r_2 + \sqrt{r_2^2 - r_0^2}}{r_0} \right) + \ln \left( \frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_0} \right) + \frac{1}{2} \left( \sqrt{\frac{r_2 - r_0}{r_2 + r_0}} + \sqrt{\frac{r_1 - r_0}{r_1 + r_0}} \right) - \frac{\sqrt{r_2^2 - r_0^2} + \sqrt{r_1^2 - r_0^2}}{2r_1} \quad (20)$$

where  $r_2$  and  $r_1$  are the radial positions of the source and observer respectively. Now define the distances from the observer to the lens and the source along the horizontal line  $D_1$  and  $D_2$  as shown in figure 1. Using

$$r_j^2 \approx D_j^2 + r_0^2 \quad \text{where } D_j \gg r_0, \quad (21)$$

we get

$$\Delta\tau/r_s = \ln \frac{4D_1 D_2}{r_0^2} + \frac{D_1 - D_2}{2D_1} - \frac{r_0}{2D} + \mathcal{O}(r_0^2/D_j^2) \quad (22)$$

where  $D$  is the reduced distance. ( $D_1$  is used in place of  $D_L$  in section 2 for handy manipulation of the index.) If we ignore the third term for now because it is usually very small, the proper time delay between two images, 1 and 2, depends only on the first term.

$$\Delta t_{12} \equiv \Delta t_1 - \Delta t_2 = -2r_s \ln(r_{01}/r_{02}). \quad (23)$$

This is exactly the same as the arrival time difference due to the logarithmic term of the time function in eq.(2) with  $z_L = 0$ . (Here the expansion of the universe is ignored.) The true time delay formula doesn't have a "quadratic term." If we recall that  $\Delta t(\text{quad}) \approx \Delta t(\text{log})$  when the source is within the Einstein radius from the lens (where the dim image is not too dim), the current standard time delay is about twice the true time delay. It is not a small difference and it makes it urgent to reanalyze the time delay data. Considering that the time delay and lens mass modelings have been producing reasonable Hubble constants while using a wrong time delay formula, it warrants special scrutinies of the fidelity of the analyses.

### 3.2 The Isotropic Schwarzschild Metric

The so-called isotropic Schwarzschild metric is of the following form,

$$ds^2 = -G(u)d\eta^2 + F(u)(du^2 + u^2 d\Omega^2), \quad (24)$$

and can be obtained from the standard Schwarzschild metric by a coordinate transformation:  $r \mapsto u$ .

$$u = \frac{1}{2} (r - r_s/2 + (r^2 - rr_s)^{1/2}); \quad (25)$$

$$F(u) = \frac{(u+a)^4}{u^4}; \quad G(u) = \frac{(u^2 - a^2)^2}{(u+a)^4}; \quad a \equiv \frac{r_s}{4} \quad (26)$$

The equations of motion are

$$\frac{d\eta}{dp} = \frac{1}{G}; \quad \frac{d\psi}{dp} = \frac{J}{r^2 F}; \quad F \left( \frac{dr}{dp} \right)^2 = \frac{1}{G} - \frac{J^2}{r^2 F}. \quad (27)$$

The deflection angle and the flight time of a photon trajectory are

$$\int d\psi = \left( \int_{u_2}^{u_0} + \int_{u_0}^{u_1} \right) \frac{1/u \, du}{\sqrt{\frac{FG_0 u^2}{GF_0 u_0^2} - 1}}; \quad (28)$$

$$\int d\eta = \left( \int_{u_2}^{u_0} + \int_{u_0}^{u_1} \right) \frac{\sqrt{F/G} \, du}{\sqrt{1 - \frac{GF_0 u_0^2}{FG_0 u^2}}}. \quad (29)$$

If one examines the deflection angle integral (17) with the factor  $r_s$  in mind, half the deflection angle comes from the term  $A$  in the numerator and the other half comes from the  $B$  term in the denominator.  $F = G^{-1}$  in the linear order, and one can see immediately that eq.(28) produces the same deflection angle as eq.(17). If we compare the time integrals eq.(18) and eq.(29), the latter has the combined factor  $G/F \approx G^2$  instead of  $B$  in the denominator and the extra factor doubles the third term in eq.(19) resulting in

$$\eta(u, u_0) = \sqrt{u^2 - u_0^2} + 2GM \ln \left( \frac{u + \sqrt{u^2 - u_0^2}}{u_0} \right) + 2GM \left( \frac{u - u_0}{u + u_0} \right)^{1/2} + \mathcal{O}(r_s^2/u_0^2). \quad (30)$$

In fact,  $\eta(u, u_0)$  can be obtained from  $t(r, r_0)$  by using eq.(25). In the linear order in  $r_s$ ,  $u = r - r_s/2$ , and the extra factor in the third term in eq.(30) can be seen coming from the first term in eq.(19).

$$(r^2 - r_0^2)^{1/2} = (u^2 - u_0^2)^{1/2} + \frac{r_s}{2} \left( \frac{u - u_0}{u + u_0} \right)^{1/2}. \quad (31)$$

So we can say that it is algebraically consistent, but it leaves a question as to which is the time the observer's clock will be measuring. As for the arrival time difference between two gravitationally lensed images, the third term is irrelevant in the linear order because it is effectively a constant.

$$2GM \left( \frac{u - u_0}{u + u_0} \right)^{1/2} = 2GM \left( 1 - \frac{u_0}{u} \right) = 2GM \quad (32)$$

where the expansion is made in  $u_0/u \ll 1$ .

Incidentally, it is worth noting that  $r_0/r$  is of the same order of  $r_s/r_0$  for bright images. They are near the Einstein ring, and hence  $r_0^2 \approx R_E^2 = 4GMD \approx 2r_s r$  (see eq.(3)). In other words,  $r_s/r = \mathcal{O}(r_s^2/r_0^2)$  for bright images. As for the gravitational lensing by the Sun as observed from the Earth, the Einstein ring radius is much smaller than the size of the Sun ( $R_E = R_\odot/23.2$ ), and the only visible "image" is highly unmagnified and  $1 \gg r_0/r \gg r_s/r_0$ .

Note that  $(r^2 - r_0^2)^{1/2}$  in eq.(31) is the distance of the straight path the photon would have flown when  $M = 0$ . So, let's reexamine eq.(17) to see if the angular span of the straight path generates an  $r_s$ -dependent extra term upon the coordinate transformation.

$$\Delta\phi(r, r_0) = \frac{\pi}{2} - \sin^{-1} \left( \frac{r_0}{r} \right) + \frac{r_s}{r_0} + \mathcal{O}(r_s^2/r_0^2),$$

The first two terms are the angular span of the straight path from  $r$  to  $r_0$ , and the second term does generate an extra term. In the linear order in  $r_s$ ,

$$\sin^{-1} \left( \frac{r_0}{r} \right) = \frac{r_0}{r} = \frac{u_0}{u} + \frac{r_s(u - u_0)}{2u^2}. \quad (33)$$

It is then curious because the extra term doesn't show up in the first order integration of eq.(28).

$$\Delta\psi(u, u_0) = \frac{\pi}{2} - \sin^{-1} \left( \frac{u_0}{u} \right) + \frac{r_s}{u_0} + \mathcal{O}(r_s^2/u_0^2). \quad (34)$$

We can reason that it is consistent because the extra term is effectively of a second order.

$$\frac{r_s(u - u_0)}{2u^2} \approx \frac{r_s}{2u} = \mathcal{O}(r_s^2/u_0^2). \quad (35)$$

We will see in a later section that the second order perturbation in  $r_s$  does generate terms of the form  $r_s^2/r_0^2$  which is of the order of  $r_s/r$  for bright images.

Upon looking at eq.(30), we may reason that the extra time is due to that  $u$  ( $u < r$ ) measures a path that is closer to the mass where the potential is deeper and so the photon speed is smaller. However, it still remains a question which formula for an observer to use between eq.(19) and eq.(30) to compare with its clock time. (The difference between the "observer"'s coordinate time and proper time is the same in the linear order in the two coordinate systems as one can see in the fourth term of eq.(20).) Given a mass  $M$ , which is rightfully considered a coordinate-invariant, we can imagine an abstract space and two abstract points  $S$  and  $O$  on it and a null geodesic connecting the two points. A coordinate system is a means to describe the abstract situation. What we are at a loss seems to be whether we are identifying the same abstract points  $S$  and  $O$  in the two well-known coordinate systems in which we examined the time delays and deflection angles. It should be worth sorting it out unambiguously.

#### 4 GEOMETRIC PATH DIFFERENCE TERM AND GRAVITATIONAL POTENTIAL TERM

Let's write the isotropic Schwarzschild metric in eq.(24) in the linear approximation as CK75 did for more general gravitational fields.

$$ds^2 = -(1 - \frac{r_s}{u})d\eta^2 + (1 + \frac{r_s}{u})(du^2 + u^2 d\Omega^2). \quad (36)$$

(CK75 uses (two times) the Newtonian potential  $-2\Phi(u)$  instead of  $r_s/u$ .) For a photon trajectory,  $ds^2 = 0$ , and the time integral can be written formally as follows.

$$\int d\eta = \int ds_3 + \int \frac{r_s}{u} ds_3, \quad (37)$$

where  $ds_3$  denotes the length element of the photon trajectory in 3-space (to distinguish from the four dimensional line element  $ds$ ). CK75 interpreted,

"The first term is the time due to the length of the path traveled and *must be computed to first order in  $G$*  [our highlight]. The second is due to the potential well through which the photon traveled."

It is intuitive and reasonable. As far as we know, this is the origin of the current standard notion of the time delay being constituted of two pieces: path difference and gravitational potential.

The question is what  $ds_3$  is, and we need to solve for the trajectory to get the accurate coordinate time delay. It is instructive to consider the straight path. If  $ds_3$  is the line length of the straight path, then  $\int ds_3 = (r^2 - r_0^2)^{1/2}$  and hence  $ds_3 = (r^2 - r_0^2)^{-1/2} r dr$ ; the integral eq.(37) reproduces the first two terms of eq.(30). The third term of eq.(30), which we discussed to be coordinate dependent and

## 6 Puzzles in Time Delay and Fermat Principle in Gravitational Lensing

irrelevant to the time delay between two images, is not recovered. For the Schwarzschild metric, we have solved the equations of motion and know that the correct line element is  $dp$ . It reproduces the time integral eq.(29 exactly in the linear order. The integral eq.(37) can be rewritten as follows in the linear order.

$$\int d\eta = \int dp + \int \frac{r_s}{u} d\ell = \int d\ell + 2r_s \left( \frac{u - u_0}{u + u_0} \right)^{1/2} + \int \frac{r_s}{u} d\ell. \quad (38)$$

where  $\ell$  is the line parameter of the straight path. We can see that  $\int dp$  generates the geometric path difference effect ( $\approx 2r_s$ ) we have seen in section 2.2 and CK75 might have liked to have.

As we have repeatedly stated, the path difference term  $2r_s$  cancels out for the arrival time difference of two images for a Schwarzschild lens. Thus for a general potential where the form of the correct line element  $dp$  may not be easily obtained, it may be reasonable to use the straight line element.

In section 2.2, we “tried to do better” and obtained the “inverse quadratic term” which we know to cancel out the true time delay. It is not any better than CK75’s doubling. As we have been repeating it, the magnitude of the “inverse quadratic term” is approximately  $2r_s$ . The fault seems to be lie in that we introduced the second order in angles while the premise is the linear order small angle approximation. CK75 erred the same. The geometric relation  $r_s/r_0 \approx r_0/r$ , where  $r$  is the position of the source or observer, seems to be the source of ready confusion. The lesson may be that one needs to be as systematic as possible. There is no obvious way to obtain the correct line element  $dp$  without solving the equations of motion, and the best bet is the straight line element. It is unlikely that CK75 would have been able to get the correct geometric path difference. However, the notion of geometric time difference is fine.

## 5 THE SECOND ORDER CORRECTIONS

The second order terms of the flight time and angular span are presented here. If we define

$$\epsilon \equiv \frac{r_s}{r}; \quad \epsilon_0 \equiv \frac{r_s}{r_0}; \quad \xi \equiv \frac{r_0}{r}, \quad (39)$$

the part of the time integral eq.(18) that is of the second order in  $r_s$  is

$$\Delta t_2(r, r_0) = \epsilon_0^2 \int \frac{3\xi^2}{2(1-\xi^2)^{1/2}} + \frac{3\xi^4(1-\xi)^2}{8(1-\xi^2)^{5/2}} dr \quad (40)$$

$$= \frac{r_s^2}{r_0} \left[ \frac{15}{8} \left( \frac{\pi}{2} - \sin^{-1} \xi \right) + \frac{(1-\xi)}{4(1-\xi^2)^{2/3}} - \frac{6-5\xi}{8(1-\xi^2)^{1/2}} \right]. \quad (41)$$

If we set  $r = \infty$ ,  $\Delta t_2(r = \infty, r_0) = 2.445 r_s^2/r_0$ . It is good to find the coefficient of the second order to be of  $\mathcal{O}(1)$ . The second order part of the deflection angle eq.(17) is

$$\Delta \phi_2(r, r_0) = \epsilon_0^2 \int \frac{\xi^2}{r_0} \left[ \frac{3\xi^2}{8(1-\xi^2)^{1/2}} + \frac{3\xi^2(1-\xi)}{4(1-\xi^2)^{3/2}} + \frac{3\xi^4(1-\xi)^2}{8(1-\xi^2)^{5/2}} \right] dr \quad (42)$$

$$= \frac{r_s^2}{r_0^2} \left[ \frac{15}{16} \left( \frac{\pi}{2} - \sin^{-1} \xi \right) + \frac{(1-\xi)}{4(1-\xi^2)^{2/3}} - \frac{6-5\xi}{8(1-\xi^2)^{1/2}} + \frac{3\xi(1-\xi^2)^{1/2}}{16} \right]. \quad (43)$$

## 6 CONCLUSION

We examined the current standard time delay formula for a Schwarzschild black hole lens in the weak field (or small angle) approximation. We find that the current standard time delay formula is wrong. It effectively doubles the true time delay. The ‘quadratic term’ is the result of a wrong calculation of the geometric path difference.

The absence of the “quadratic term” thwarts the claim of the relation between the time delay and the lens equation through the Fermat’s principle. However, the CSTD formula can be considered a generating function of the lens equations because it works, even though it is unclear presently what significance it has.

We call to reanalyze the strong gravitational lenses with time delay measurements. We expect the correct time delay formula to help for better lens modelings. On the other hand, the current practices of reasonably good fittings to the Hubble constant while using a wrong time delay formula makes one worried lest the data analyses tend to converge on desirable values. We call for exceptional scrutinies of the fidelity of the analysis methods.

We identified a term in time delay that is dependent on the coordinate systems, even though it is benign for the arrival time difference of two images. The coordinate dependence of the “measurables” seems to require some careful thoughts as to what is what we measure.

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